LECTURE 14 IMPLICIT DIFFERENTIATION

Last lecture, we finished the Chain Rule and started implicit differentiation, as a direct application of the Chain Rule. Today, we focus on more problems involving implicit differentiation. In this section, you need basic knowledge such as the Power Chain Rule,

$$\frac{d}{dx}g\left(x\right)^{n}=ng\left(x\right)^{n-1}g'\left(x\right)=ng\left(x\right)^{n-1}\frac{dg}{dx},$$

or general Chain rule

$$\frac{d}{dx}f\left(g\left(x\right)\right)=f'\left(g\left(x\right)\right)g'\left(x\right).$$

Example 1. Find the slope of the circle $x^2+y^2=25$ at the point $(3,-4)$.

Solution. The circle cannot be written as a function, but as a combination of two functions. Why? The circle does not pass the vertical line test. We have the upper semicircle

$$y_1=\sqrt{25-x^2},\quad -5\leq x\leq 5,$$

(see that it is always positive) and the lower semicircle

$$y_2=-\sqrt{25-x^2},\quad -5\leq x\leq 5.$$

Clearly, $(3,-4)$ belongs to the lower semicircle. Therefore, by the Chain Rule,

$$\frac{dy_2}{dx}\mid_{x=3}=-\frac{-2x}{2\sqrt{25-x^2}}\mid_{x=3}=\frac{3}{4}.$$

The problem is MUCH simpler using implicit differentiation on the original equation.

$$\frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}\left(25\right)\implies 2x+2y\frac{dy}{dx}=0\implies\frac{dy}{dx}=-\frac{x}{y}.$$

Thus,

$$\frac{dy}{dx}\mid_{x=3,y=-4}=\frac{3}{4}.$$

In fact, $\frac{dy}{dx}=-\frac{x}{y}$ works for every points on the circle (so no need to chop into upper and lower semicircles) except for $y=0$.

Remark. Differentiate both sides of the equation with respect to x, treating $y=y\left(x\right)$ as a differentiable function of x. Then, solve for $\frac{dy}{dx}$.

Let's see more involved functions.

Example 2. Find $\frac{dy}{dx}$ when $x^3+y^3=18xy$.

Solution. Differentiate both sides with respect to x.

$$\frac{d}{dx}\left(x^3+y^3\right)=18\frac{d}{dx}\left(xy\right)$$

which yields

$$\begin{aligned} 3x^2 + 3y^2\frac{dy}{dx} &= 18\left(y + x\frac{dy}{dx}\right) \\ \implies 3x^2 + 3y^2\frac{dy}{dx} &= 18y + 18x\frac{dy}{dx} \\ \implies \frac{dy}{dx}\left(3y^2 - 18x\right) &= 18y - 3x^2 \\ \implies \frac{dy}{dx} &= \frac{18y - 3x^2}{3y^2 - 18x} \\ &= \frac{6y - x^2}{y^2 - 6x}. \end{aligned}$$

Remark. You may ask, what happened here with $\frac{d}{dx}(xy)$? Why is it $y + x\frac{dy}{dx}$? Here is the clue. Let $f(x) = x$, and also think of $y = y(x)$ as a function of x. Then, $\frac{d}{dx}(xy) = \frac{d}{dx}(f(x)\,y(x))$. By product rule, we have

$$\frac{d}{dx}(f(x)\,y(x)) = f'(x)\,y(x) + f(x)\,y'(x).$$

Certainly, $f'(x) = 1$ since $f(x) = x$. Therefore,

$$\frac{d}{dx}(xy) = \frac{d}{dx}(f(x)\,y(x)) = f'(x)\,y(x) + f(x)\,y'(x) = y(x) + xy'(x) = y + x\frac{dy}{dx}.$$

One could also ask for more order of derivatives.

Example 3. Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$.

Solution. We find the first derivative first.

$$\begin{aligned} \frac{d}{dx}\left(2x^3 - 3y^2\right) &= \frac{d}{dx}(8) \\ \implies 6x^2 - 6y\frac{dy}{dx} &= 0 \\ \implies \frac{dy}{dx} &= \frac{x^2}{y}, \quad y \neq 0. \end{aligned}$$

Now, to find the second derivative, we differentiate both sides again,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{x^2}{y}\right) \\ &= \frac{2xy - x^2\frac{dy}{dx}}{y^2} \\ &= \frac{2xy - x^2\frac{x^2}{y}}{y^2} \\ &= \frac{2x}{y} - \frac{x^4}{y^3}. \end{aligned}$$

Remark. How is this useful? Think of x as time, and y as displacement. Then the original equation describes the displacement of an object. What you found as second derivative is the acceleration, that is, when multiplied by the object's mass, you get a representation of Newton's second law. In some sense, you can build a rocket with propulsion forces according to the second derivative, it will give you the whacky curve describing the motion of the object, given by $2x^3 - 3y^2 = 8$.

Whenever you find the derivative, there will always be a question for the equation of the tangent line, even the equation of the normal (perpendicular to the tangent line, at the point of tangency).

Example 4. Consider $x^2 + xy - y^2 = 1$. Find the equation of the tangent line and normal to the curve at $(2, 3)$.

Solution. To find the slope of the tangent line at $(2,3)$, we need $\frac{dy}{dx}$ and then evaluate at $(2,3)$.

$$\begin{aligned} \frac{d}{dx}\left(x^2+xy-y^2\right) &= \frac{d}{dx}(1) \\ \implies 2x+\left(y+x\frac{dy}{dx}-2y\frac{dy}{dx}\right) &= 0 \\ \implies 2x+y+(x-2y)\frac{dy}{dx} &= 0 \\ \implies \frac{dy}{dx} &= -\frac{2x+y}{x-2y}. \end{aligned}$$

Therefore, the slope at $(2,3)$ is

$$m=\frac{dy}{dx}\,|_{(2,3)}=-\frac{2\times 2+3}{2-2\times 3}=\frac{7}{4}.$$

The equation for the tangent line then is

$$y-3=\frac{7}{4}(x-2) \implies y=\frac{7}{4}x-\frac{1}{2}.$$

The normal is perpendicular to the tangent line at the point of tangency, thus sharing the same point $(2,3)$. Its slope is the negative reciprocal of the slope of the tangent, which is $-\frac{4}{7}$. Thus, the normal has equation

$$y-3=-\frac{4}{7}(x-2) \implies y=-\frac{4}{7}x+\frac{29}{7}.$$